

Graph theory

Many pure mathematicians all over the world have, no doubt at some point, been exposed to some elementary graph theory. Since it is a *pure* mathematics course, the focus would very likely have been on basic concepts, basic theorems, and then on not-so-basic concepts and theorems. The focus of this article, however, is to shed some light on the often overlooked areas in graph theory - its amazingly diverse applications and a subset called networks.

We begin with the first graph theory paper to be published - by Euler - on the Seven Bridges of Königsberg (now Kaliningrad in Russia). The city was built on the Pregal River which divided it into two banks and two islands, flowing under seven bridges, and the question posed by its citizens was whether one could take a walk that crosses each bridge exactly once and return to the beginning. Euler reconstructed the puzzle into a graph problem and found that there, in fact, was no solution. To describe and understand the graph he used, we need to introduce a few concepts from graph theory itself.

First, we have the bread and butter, nodes and edges. Nodes, also known as vertices are often used to denote the concrete objects in models, e.g. here we have the land masses as the node of Euler's proposed graph. Edges are then used to denote the relationship of between the nodes, thus here we have the bridges to connect the land masses as edges connecting the nodes. The final terminology we now need to explain is the aptly named Eulerian circuit, which exists in a graph if you can traverse each edge of the graph exactly once and finish where it started.

Now since Euler's Seven Bridges graph is a connected one (as you can always access a particular land mass via a bridge), there is a theorem that says that a graph has an Eulerian circuit if and only if all of its nodes have even degree. The degree of a node is simply the total number of edges the node has. So, taking a look at the graph (pic), we see that all the nodes have degree three, confirming Euler's certainty. To put the simplicity of this solution into perspective, the theorem mentioned is exactly one of the very first 'basic theorems' one learns and is shorter than if this article were to end now!

As one may suspect not all graph theory problems are as easily solved as the example I have shown. A related and similar-sounding construction to Eulerian circuits are Hamiltonian cycles, named after Sir William Rowan Hamilton (founder of quaternions). A Hamiltonian cycle can be seen as a sort of 'inverse' to an Eulerian circuit in that it exists if one can traverse each **node** of the graph once and finish where it started. Contrary to expectations however, an efficient and reliable method of determining whether a graph has a Hamiltonian cycle or not has not been found (yet).

Whilst on the topic of nodes, an important and often misrepresented problem

is vertex-colouring. The most infamous one being the Four-Colour Theorem - where it states that any map in a plane can be coloured using four colours in such a way that regions sharing a common boundary (other than a single point) do not share the same colour. This was the first major theorem to be proven using a computer and is not without its controversy, with many mathematicians today refusing to accept this 'proof by exhaustion'.

1 Intro

- Constraint Satisfaction Problem - e.g. timetabling and 4 colour thm, expand on later

2 Applied graphs

- Apps to Biology & computer
- Networks
 - (Online) Social networks
 - Brain