Definitions for AGTA

Samantha Lam

- A **(finite) strategic form game** Γ, with n players, consists of
  1. A set $N = \{1, \ldots, n\}$ players.
  2. For each $i \in N$, a (finite) set $S_i = \{1, \ldots, m_i\}$ of (pure) strategies, and $S = S_1 \times \ldots \times S_n$ is the set of all possible combinations of (pure) strategies.
  3. For each $i \in N$, a utility/payoff function $u_i : S \to \mathbb{R}$ describes the payoff $u_i(s_1, , s_n)$ to player i under each combination of stategies.
- A **zero-sum game** Γ is one where for all $s = (s_1, \ldots, s_n) \in S$, $\sum_{i=1}^{n} u_i(s) = 0$.
- A **mixed/randomised strategy** x_i for player i is a probability distribution over $S_i = \{1, \ldots, m_i\}$. I.e. x_i is a vector such that $x_i = \{x_i(1), \ldots, x_i(m_i)\}$,such that $x_i(j) \geq 0$ for all $j \in S_i$ and $\sum_{j=1}^{m_i} x_i(j) = 1$.

- $x = (x_1, \ldots, x_n) \in X = X_1 \times \ldots \times X_n$ is a profile/set of all combinations of mixed strategies.

- $s = (s_1, \ldots, s_n) \in S = S_1 \times \ldots \times S_n$ is a combination of pure strategies.

- $x(s) = \prod_{j=1}^{n} x_j(s_j)$ = probability of combination s under mixed profile x.

- The **expected payoff** of player i under a mixed profile $x = \{x_1, \ldots, x_n\} \in X$ is $U_i(x) = \sum_{s \in S} x(s) * u_i(s)$.
- A mixed strategy $x_i \in X$ is a **pure strategy** if $x_i(j) = 1$ for some $j \in S_i$ and $x_i(j') = 0$, for all $j' \neq j$. Such a strategy is denoted $\pi_{i,j}$, i.e. player i chooses with probability 1 strategy j.

- $(x_{-i}; y_i) = \{x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n\}$ is where everybody's strategy remains the same as the original profile $(x_1, \ldots, x_n)$ except for player i who switches mixed strategy from x_i to y_i.

- A mixed strategy $z_i \in X_i$ is the **best response** for player i to x_i if $\forall y_i \in X_i$, $U_i(x_{-i}; y_i) \leq U_(x_{-i}; z_i)$.
- For a strategic game Γ, a strategy profile $x = (x_1, \ldots, x_n) \in X$ is a **mixed Nash Equilibrium** if, for every player i, x_i is the best response for x_{-i}.
  I.e. for every player $i = 1, \ldots, n$, and every mixed strategy $y_i \in X_i$,
  $U_i(x_{-i}; x_i) \geq U_(x_{-i}; y_i)$,
  i.e. no player can improve its payoff by unilaterally deviating from the mixed strategy profile $x = (x_1, \ldots, x_n)$.

* x is a **pure Nash Equil.** if every x_i is a pure strategy $\pi_{i,j}$ for some $j \in S_i$.

- **Nash's Theorem** states that every finite, n-person strategic game has a mixed Nash Equilibrium.

- A finite **2p-zs** strategic game Γ is one where for players $i \in 1, 2$, the payoff functions $u_i : S \to \mathbb{R}$ are such that for all $s = (s_1, s_2) \in S$, $u_1(s) + u_2(s) = 0$.

- $x_1^* \in X_1$ is a **minmaximiser** for player 1 if
$\min_{x_2 \in X_2}((x_1^*)^T A x_2) = \max_{x_1 \in X_1} \min_{x_2 \in X_2}(x_1^T A x_2)$.

- $x_2^* \in X_2$ is a **maxminimiser** for player 2 if
$\max_{x_1 \in X_1}(x_1^T A x_2^*) = \min_{x_2 \in X_2} \max_{x_1 \in X_1}(x_1^T A x_2)$.

- **Minimax Theorem**: Let a 2p-zs game Γ be given by an $m_1 \times m_2$ matrix, A, of real numbers. Then there exists a unique $v^* \in \mathbb{R}$ such that $\exists x^* = (x_1^*, x_2^*) \in X$ such that:

  1. $((x_1^*)^T A)_j \geq v^*$, for $j = 1, \ldots, m_2$.
  2. $(A x_2^*)_i \leq v^*$, for $i = 1, \ldots, m_1$.
  3. Thus, $v^* = (x_1^*)^T A x_2^*$, and
  $\max_{x_1 \in X_1} \min_{x_2 \in X_2}(x_1^T A x_2) = v^* = \min_{x_2 \in X_2} \max_{x_1 \in X_1}(x_1^T A x_2)$.
  4. Conditions 1-3 hold precisely when $x^* = (x_1^*, x_2^*)$ is *any* Nash Equilibrium.

- For $x_i, x_i' \in X_i$, x_i **dominates** x_i', $x_i \succeq x_i'$, if, for all $x_{-i} \in X_{-i}$,
$U_i(x_{-i}; x_i) \geq U_i(x_{-i}; x_i')$.

- For $x_i, x_i' \in X_i$, x_i **strictly dominates** x_i', $x_i \succ x_i'$, if, for all $x_{-i} \in X_{-i}$, $U_i(x_{-i}; x_i) > U_i(x_{-i}; x_i')$.

- A mixed strategy $x_i \in X_i$ is **dominant** if, $\forall\, x_i' \in X_i, x_i \succeq x_i'$.

- **support(x_i)** for a mixed strategy x_i is the set of pure strategies $\pi_{i,j}$ such that $x_i(j) \geq 0$.

- A strategy $x_i \in X_i$ is **strictly dominated** if there exists some $x_i' \in X_i$ such that $x_i' \succ x_i$.

- A strategy $x_i \in X_i$ is **weakly dominated** if there exists some $x_i' \in X_i$ such that $x_i' \succeq x_i$, and for some $x_{-i} \in X_{-i}$, $U_i(x_{-i}; x_i') > U_i(x_{-i}; x_i)$.

- **Rational Knowledge (RKN) hypothesis** is such that every player's rationality (which is to maximise its own expected payoff is common knowledge among all players. *Ratinal Common Knowledge (RCK)* assumption is such that every player is rational, and every player knows that every player is rational, and every player knows that every player knows that every player is rational, etc, for every 'depth' of knowledge.
Thus, it justifies the iterated elimination of strictly dominated strategies because the reasoning used to do this mimics the reasoning of a rational player who makes the RCK assumption.

- Iterative elimination of strictly dominated strategies:
While(some pure strategy $\pi_{i,j}$ is strictly dominated)
eliminate $\pi_{i,j}$ from the game
obtain a new residual game
Repeat this procedure and the resulting payoff is a pure strategy (which is a NE).
Else, a residual game can be solved with mixed strategies (maybe).

- A **Linear Programming/Optimization (LP)** problem instance (f, Opt, C) consists of

1. A linear *objective function* $f : \mathbb{R}^n \to \mathbb{R}$, given by:
   $f(x_1, \ldots, x_n) = c_1x_1 + \ldots + c_nx_n + d$,
   where the coefficients c_i and d are assumed to be constant, rational numbers.
2. An *optimization criterion*: $Opt \in \{Maximize, Minimize\}$.
3. A set/system $C(x_1, \ldots, x_n)$ of m linear constraints (i.e inequalities/equalities), $C_i(x_1, \ldots, x_n),\ i = 1, \ldots, m$, where each $C_i(x)$ has the form:
   $a_{i,1}x_1 + \ldots + a_{i,n}x_n \Delta b_i$, where $\Delta \in \{\leq, \geq, =\}$, $a_{i,j}, b_i \in \mathbb{Q}$.

- An **optimal feasible solution** is one where $x^* \in \mathbb{R}^n$ is a solution to the constraints of the LP and x^* maximises/minimises the objective, i.e. for any other solution x' satisfying all constraints $c^T x^* \geq / \leq c^T x'$.
- An LP is said to be **feasible** if the set of all solutions to the system $C(x),\ K(C) \subseteq \mathbb{R}^n$, is non-empty. (A vector $v \in \mathbb{R}^n$ is a solution to the system $C(x)$ if v satisfies every constraint $C_i \in C$, i.e. $C_1(v) \wedge \ldots \wedge C_n(v)$ holds true.)
- An **optimal solution** for $Opt = Maximize$ ($Minimize$), is some $x^* \in K(C)$ such that $f(x^*) = \max_{x \in K(C)}(f(x))$ ($\min_{x \in K(C)}(f(x))$)
- Given an LP problem instance (f, Opt, C) as input, the output is one of the following 3:
  1. The problem is Infeasible.
  2. The problem is Feasible but Unbounded.
  3. An Optimal Feasible Solution (OFS) exists. One such optimal solution is $x^* \in \mathbb{R}^n$ and the optimal objective value is $f(x^*) \in \mathbb{R}^n$.

- To convert LP into a **dictionary**, add variables $y_j \geq 0, j = 1, \ldots, m$, so that it the $\leq$ becomes $=$.
- This dictionary is a **feasible dictionary** if each $b_j \geq 0, j = 1, \ldots, m$, and the **feasible solution** is $x_i = 0, y_j = b_j, i = 1, \ldots, n, j = 1, \ldots, m$.
- $f(0) = d$ is **basic feasible solution (BFS)** with basis $B = \{y_1, \ldots, y_m\}$

- **Primal LP**: Maximise $c^T x$
  Subject to:
  $Ax \leq b,\ x_1, \ldots, x_n \geq 0$.
- **Dual LP**: Minimize $b^T y$
  Subject to:
  $A^T y \geq c,\ y_1, \ldots, y_m \geq 0$.
- **Strong Duality, von Neumann47** One of the following four situations hold:
  1. Both the primal and dual LP are feasible, and for the optimal solutions x^* of the primal and y^* of the dual: $c^T x^* = b^T y^*$.
  2. The primal is infeasible and the dual is unbounded.
  3. The primal is unbounded and the dual is infeasible.
  4. Both LPs are infeasible.
- A **memoryless strategy** for some player i is a strategy s_i which takes some action, i.e. moves across the same edge, out of each vertex v belonging to player i, regardless of the history of the game.

- **Kuhn's Thm53**: every finite n-person extensive Perfect Information game (i.e. one node per information set), $\mathcal{G}$, has a Nash Equilibrium in pure strategies. I.e Some pure profile, $s^* = \{s_1^*, \ldots, s_n^*\}$, is a Nash Equilibrium.